General Synthesis of Quarter-Wave Impedance Transformers*

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Summary—This paper presents the general synthesis of a radio frequency impedance transformer of n quarter-wave[1] steps, given an "insertion loss function" of permissible form. This procedure parallels that of Darlington for lumped constant filters by providing the connection between Collin's canonical form for the insertion loss function and Richards' demonstration that a reactance function may always be realized as a cascade of equal length impedance transformers terminated in either a short or open circuit. In particular, it is shown that insertion loss functions of the form selected by Collin are always realizable with positive characteristic impedances, and that the synthesis procedure, for maximally flat and Tchebycheff performance, involves the solution, at most, of quadratic equations. In addition, this procedure permits the proof of Collin's conjecture that, for his insertion loss function, the resulting reflection coefficients are symmetrical. Finally, closed expressions are given for the coefficients of the input impedance of a given n section transformer in terms of the n characteristic impedances and vice versa.

Introduction

POSSIBLY the most frequent problem encountered in the design of distributed constant high frequency transmission line circuits, is that depicted in Fig. 1, where the several unknown characteristic

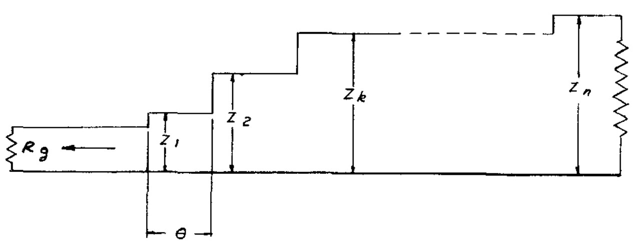

Fig. 1—Schematic of n section transformer.

impedances are to be chosen so as to minimize the input vswr over a given band of frequencies. The exact solution for a single step transformer is given by the well-known relationship $Z_1 = \sqrt{R}$. This result has found wide practical application. Nevertheless, in the range of microwave frequencies there are many problems which require more elaborate transformer designs. It is only recently that Collin[2] has given exact solutions for the cases of two, three, and four transformer sections. Previously the designer had available only approximate solutions depending on the assumption that the individual reflections are so small that multiple reflections can be neglected.

It is on this basis that the well known solution, that the logarithms of the ratios of the characteristic impedances vary according to the binomial coefficients, may be derived. Recently Cohn[3] has used the same approximation to obtain Tchebycheff performance in the band of low vswr. The same idea was applied by Riblet[4] to the equivalent directional coupler problem.

Although approximate solutions are easier to use and may be adequate for most applications, they cannot, in general, provide limits on optimum performance and do not provide a satisfactory foundation for a deeper understanding of the problem. For example, this exact synthesis procedure will prove certain general results which were only conjectured by Collin and not even suggested by those concerned with approximate solutions.

Now Collin's exact solution for cases $n=2$, 3, 4 is not a true synthesis, but rather a solution by the method of undetermined coefficients. He determines the insertion loss function of a general transformer containing n unknown characteristic impedances by equating the unknown general insertion loss function to a special insertion loss function having the desired performance. The procedure leads, in general, to the simultaneous solution of higher degree algebraic equations each containing some or all of the unknowns. Disregarding the numerical difficulties involved, the process leaves a number of questions unanswered. Of these, the most important is the question of physical realizability, since Collin's procedure gives no assurance that the characteristic impedances obtained from the simultaneous algebraic equations will be positive real numbers. Without this assurance, no claim for optimum performance is justified.

* Manuscript received by the PGMTT May 18, 1956. Presented before the IRE West Coast Convention, August 1955, San Francisco, Calif. Since this paper was first submitted to the PROC. IRE for possible publication, the writer has learned of three other papers on the filter aspects of this theory. Why the treatment of physical realizability in each of these is incomplete, is discussed at the end of this paper.

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[1] This discussion is in no way limited to quarter-wave transformers, since the only assumption required is that the steps be of the same length. This special case has been chosen, however, because the use of quarter-wave transformers rules out the possibility of super match, and because it is very likely that the optimum general monotonic transformer has quarter-wave steps, since this has already been demonstrated by the writer in the limiting case of narrow bandwidth and small impedance transformation, with the assistance of a key theorem proved for the purpose by J. E. Eaton.

[2] R. E. Collin, "Theory and design of wide-band multisection quarter-wave transformers," PROC. IRE, vol. 43, pp. 179–185; February, 1955.

[3] Seymour B. Cohn, "Optimum design of stepped transmission-line transformers," IRE TRANS., vol. MTT-3, pp. 16–21; April, 1955.

[4] H. J. Riblet, "Super directivity with directional coupler arrays," PROC. IRE, vol. 40, pp. 994–995; August, 1952.

Now the key to physical realizability is provided by a theorem due to Richards,[5] which he has applied to the realization of reactance functions as a cascade of impedance transformers terminated in an open or short circuit.[6] In order to extend this synthesis procedure to a cascade of impedance transformers terminated in a resistance, it will be necessary and sufficient to impose an additional condition on the class of allowable impedance functions, and it will be shown that this condition is consistent with Collins' canonical insertion loss function so that one may proceed from the given insertion loss function to a step by step reconstruction of the network as a cascade of impedance transformers terminated in a resistance.

The Problem

Consider a section of uniform lossless transmission line of characteristic impedance Z_c, phase constant $2\pi/\lambda_g$, and length l. Then it is readily shown that the voltage and current at the input to the line section are linearly related to the voltage and current at the output by the equations

$$v_i = \cos\theta v_0 + jZ_c \sin\theta i_0$$

$$i_i = j \sin v_0 + \cos\theta i_0$$

with $\theta = 2\pi l/\lambda g$. In vector notation this is written

$$\begin{pmatrix} v_i \\ i_i \end{pmatrix} = \begin{bmatrix} \cos\theta & j\sin\theta Z_c \\ \dfrac{j\sin\theta}{Z_c} & \cos\theta \end{bmatrix} \cdot \begin{pmatrix} v_0 \\ i_0 \end{pmatrix}. \qquad (1)$$

The matrix appearing above will be called the impedance matrix. The elements appearing in it are the general circuit parameters; and its determinant, as a consequence of reciprocity, is equal to unity. If the line section is terminated in an impedance Z_L so that $v_0 = Z_L i_0$, the input impedance, Z_i is readily found from (1). The impedance matrix of a cascade of line sections, as is shown in Fig. 1, is obtained by multiplication of the impedance matrices of the individual line sections. It is readily shown by induction that the impedance matrix of a cascade of n sections will have the form,

$$\begin{bmatrix} a_0 \cos^n\theta + a_2 \cos^{n-2}\theta + \cdots & j\sin\theta(a_1 \cos^{n-1}\theta + a_3 \cos^{n-3}\theta + \cdots) \\ j\sin\theta(c_1 \cos^{n-1}\theta + c_3 \cos^{n-3}\theta + \cdots) & c_0 \cos^n\theta + c_2 \cos^{n-2}\theta + \cdots \end{bmatrix} \qquad (2)$$

with leading coefficients a_0, a_1, c_0, c_1 all real and greater than zero. By a well-known theorem in the theory of determinants, it is known that the determinant of (2) is again unity. The input impedance of such a cascade, for an arbitrary termination R is obtained as above by multiplying elements of the first column by R and dividing the sum of the resulting terms in the first row by the sum of those in the second.

It will be convenient, numerically, to reverse the procedure by constructing the impedance matrix from a known input impedance. This requires the identification of R, and the restoration of any constant multiplicative factor which may have been cancelled out. Both of these questions are resolved by making use of the fact that the impedance matrix (2) must reduce to the unit matrix for $\cos\theta = 1$.

The general theorems of analytic function theory are made available to us by the introduction of the frequency variable $p = -j\cos\theta/\sin\theta$.[7] Now as the guide wavelength λ_g varies from zero to infinity, *i.e.*, over all positive real frequencies, p ranges over all negative and positive values infinitely often. By theorems on analytic continuation, any formal result which is proved for a single determination of p will be valid for all the others. Thus we may treat p as a frequency variable in the familiar sense without any limitation on bandwidth. With this in mind, we may write

$$\begin{bmatrix} \cos\theta & j\sin\theta Z_c \\ \dfrac{j\sin\theta}{Z_c} & \cos\theta \end{bmatrix} = j\sin\theta \begin{bmatrix} p & Z_c \\ \dfrac{1}{Z_c} & p \end{bmatrix}. \qquad (3)$$

Thus the impedance transformation for a single section of line may be written,

$$Z_i = Z_c \frac{pZ_L + Z_c}{Z_L + pZ_c} \qquad (4)$$

where Z_L is the impedance terminating the transformer section. It is clear that the term $j\sin\theta$ appearing in (3) cancels out in (4). The same is true for the input impedance of a cascade of such transformers. The general input impedance of a cascade of such transformer sections, terminated in R can readily be shown to be

$$Z_i = \frac{(p^n + \sigma_2^e p^{n-2} + \cdots)R + (\sigma_1^e p^{n-1} + \sigma_3^e p^{n-3} + \cdots)}{\sigma_1^0 p^{n-1} + \sigma_3^0 p^{n-3} + \cdots)R + (p^n + \sigma_2^0 p^{n-2} + \cdots)} \qquad (5)$$

with the σ's all real and positive.

If we consider complex as well as imaginary values of p and write $p = \sigma + iw$ and consider values of Z_i corresponding to values of p for which $\sigma \geqq 0$, we find by simple calculation that, whenever the real part of Z_L is positive, so is the real part of Z_i. In the terminology of Brune,[8] Z_i is a positive real function of p, whenever Z_L is positive real. We thus find, as a necessary condi-

[5] P. I. Richards, "A special class of functions with positive real part in a half-plane," *Duke Math. J.*, vol. 14, pp. 777–786, th. 6; September, 1947. See p. 779.

[6] P. I. Richards, "Resistor-transmission-line circuits," Proc. IRE, vol. 34, pp. 217–220; September, 1946. See p. 219.

[7] This differs from the frequency variable chosen by Richards and was selected so that $p = 0$ for transformer sections one-quarter wavelength long.

[8] O. Brune, "Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency," *J. of Math. and Phys.*, vol. 10, pp. 191–236; October, 1931.

tion, that all impedance functions of p, of the form in (5), realizable as a cascade of transformer sections terminated in a resistance, must be positive real functions of p. Furthermore, we observe that the determinant of (3), neglecting $j \sin \theta$, is equal to p^2-1. Consequently, the product of the two polynomials in (5) beginning with p^n minus the product of the remaining polynomials is necessarily equal to $R(p^2-1)^n$. This theorem forms the basis for the synthesis procedure and may be summarized as described below.

Synthesis Theorem

The necessary and sufficient conditions that a rational function of p with real coefficients, written in the form

$$Z = \frac{m_1(p) + n_1(p)}{m_2(p) + n_2(p)}$$

with m_1 and m_2 odd or even and n_1 and n_2 even or odd, be the input impedance of a cascade of n, equal-length transmission line sections terminated in a resistance are: 1) Z must be a positive real function of p, and 2) $m_1(p)m_2(p) - n_1(p)n_2(p) = C(p^2-1)^n$. That these conditions are necessary has already been proven; that they are also sufficient is shown in an appendix.

Now it is well known that the insertion loss function P_L giving the ratio of the power available from a generator of internal impedance R_g to that dissipated in a termination of impedance R_L, through a network having the general circuit parameters A, B, C, D is

$$P_L = \frac{|AR_L + DR_g + B + CR_gR_L|^2}{4R_gR_L}. \tag{6}$$

We thus immediately infer from (2) that the insertion loss function can always be expressed as an even real polynomial in $\cos\theta$. This is Collin's theorem.

If we write the power loss ratio as

$$P_L = 1 + P_n(\cos^2\theta) \tag{7}$$

where P_n is a real polynomial of nth degree, we may attempt to select P_n so that all of its zeros fall in the frequency band of low vswr. This will give the maximum number of frequencies of zero loss. To avoid values of P_L less than 1, these roots must be double and we are led to the form,

$$P_L = 1 + Q_n^2(\cos\theta) \tag{8}$$

where Q_n is even or odd in $\cos\theta$. Collin has chosen $Q_n(\cos\theta) = kTn(\cos\theta/s)$ for Tchebycheff performance and $k \cos^n\theta$ for maximally flat performance, but the conclusions to follow will be true for any power loss ratio of the form (8).

Our problem then is to reverse the procedure above. We will start with a given power loss ratio of the form (8) and show how one may proceed step by step to determine the characteristic impedances Z_i which will result in the required power loss ratio. The numerical work is carried through most conveniently with the impedance matrices of the form of (2). Proof of physical realizability in terms of positive characteristic impedances will require the use of the synthesis theorem.

Along the way, we shall prove Collin's conjecture that $Z_iZ_{n+1-i} = R$ and derive closed expressions for the σ's of (5) in terms of the Z_i's, and, conversely, express the Z_i's explicitly in terms of the a's and c's of (2).

Solution of Problem

Now

$$P_L = \frac{1}{1 - |\Gamma|^2}$$

where $|\Gamma|^2$ is the square of the magnitude of the input reflection coefficient. Thus

$$|\Gamma|^2 = \frac{Q_n^2(\cos\theta)}{1 + Q_n^2(\cos\theta)}. \tag{9}$$

When $\cos^2\theta$ is replaced in (9) by p^2/p^2-1, condition 1[9] of the basic theorem, plus the requirement that Γ have the proper value at infinite frequency, yields a unique determination for Γ. Consider $\Gamma = Z-1/Z+1$. Since Z is to be pr, $|\Gamma| \leqq 1$ for all values of p in the right half plane. In particular, Γ can have no poles in the right half plane. Consider the zeros of $1+Q_n^2(\cos\theta)$. We need be concerned only with the n zeros of the expression as a function of $\cos^2\theta$. Now $p^2 = \cos^2\theta/(\cos^2\theta - 1)$. Thus to each $\cos^2\theta$ root there exist two p roots differing in argument by 180°. Of these, one must fall in the left half plane except for the possibility that the two roots are imaginary. This is contrary to our choice of the power loss ratio, because it implies actual frequencies at which the power loss ratio is zero. Thus we select the n p-roots which fall in the left half plane, and use them to construct the denominator of Γ. When p is replaced by $-j\cos\theta/\sin\theta$, it will be found that

$$\Gamma = \frac{\alpha Q_n(\cos\theta)}{\cos^n\theta + \cdots + j\sin\theta(\cos^{n-1}\theta + \cdots)}. \tag{10}$$

The value of the complex constant α is uniquely determined from the requirement that

$$\Gamma = \frac{R-1}{R+1} \text{ for } \cos\theta = 1.$$

If Γ is written in p form the same determination can be made by putting $p = \infty$. Having constructed Γ, the input impedance Z is immediately determined from the

[9] For a discussion of the consequences of this assumption, the reader is referred to E. A. Guillemin, "The Mathematics of Circuit Analysis," John Wiley and Sons, Inc., New York, N. Y., 1949. See ch. 6, arts. 26–27.

equation

$$Z = \frac{1 + \Gamma}{1 - \Gamma}. \quad (11)$$

We must show that our construction of Γ, to avoid poles in the right half plane, yields a function which satisfies the pr condition of the theorem. Now, since Γ is analytic in the right half plane, according to the maximum modulus theorem, the maximum absolute value which it can achieve in the right half plane is attained on the imaginary axis. By construction, however, its value here is given by (9) and thus $|\Gamma| \leqq 1$ for all values of p in the right half plane. From (1) the real part of Z is readily found to be $(1-|\Gamma|^2)|1-\Gamma|^2$ and thus Z is pr.

It remains to show that the Z so constructed must satisfy condition 2 of the synthesis theorem. If the $\cos^2\theta$ in (9) is replaced by p^2/p^2-1, $|\Gamma|^2$ can be written

$$|\Gamma|^2 = \frac{R_n(p^2)}{(p^2 - 1)^n + R_n(p^2)}. \quad (12)$$

Now the Z obtained from (10) may be written in the form

$$Z = \frac{m_1(p) + n_1(p)}{m_2(p) + n_2(p)} \quad (13)$$

where m_1 and m_2 are even or odd and n_1 and n_2 are odd or even. Since Z is positive real, it is well known[10] that all the coefficients appearing in Z are of the same sign, and the degree of the numerator and denominator differ at most by unity. From (13), Γ becomes

$$\Gamma = \frac{m_1 - m_2 + (n_1 - n_2)}{m_1 + m_2 + (n_1 + n_2)}. \quad (14)$$

Now since $p = -j\cos\theta/\sin\theta$, $|\Gamma|^2$ is

$$|\Gamma|^2 = \frac{(m_1 - m_2)^2 - (n_1 - n_2)^2}{(m_1 + m_2)^2 - (n_1 + n_2)^2}.$$

The difference between the denominator and numerator of this fraction is $4m_1(p)m_2(p) - 4n_1(p)n_2(p)$. Comparing this with (12) we see that

$$m_1(p)m_2(p) - n_1(p)n_2(p) = Cp^2 - 1)^n.$$

Thus condition 2 is satisfied by our construction. It is also readily argued that the numerator and denominator of (13) must be of the same degree.

Once Z is known, the first unknown characteristic impedance is readily determined by subjecting Z to the transformation inverse to (4) with Z_c selected to have the value of Z for $p=1$, so that the degree of the resulting impedance function is one less than the degree of Z. This procedure can then be repeated, and the synthesis theorem assures us that all of the characteristic impedances so found are real, positive numbers.

[10] *Ibid.*, pp. 396–411.

For the purpose of numerical computation, it appears to be a little more convenient to use the impedance matrix in the form of (2) which is obtained by substituting (10) in (11). The load resistance R and proper multiplicative constant for the numerator and denominator are obtained from the condition that (2) must reduce to the unit matrix for $\cos\theta = 1$. If we multiply (2), either on the left or the right, by the inverse of (1)

$$\begin{pmatrix} \cos\theta & -j\sin\theta Z_c \\ \dfrac{-j\sin\theta}{Z_c} & \cos\theta \end{pmatrix}$$

the characteristic impedance, of either the first or last sections of the transformer may be determined from the condition that the resulting matrix must have elements whose degrees in $\cos\theta$ are reduced by one.

Special Results

We are now in a position to prove Collin's conjecture that, for power loss ratios of the form

$$P_L = 1 + Q_n^2(\cos\theta)$$

the impedances Z_i will satisfy the relationship $Z_iZ_{n+1-i} = R$. Since the numerator of $|\Gamma|^2$ is a perfect square, involving only $\cos\theta$, [see (9)], Γ has the form of (10). Since Γ is real for $\cos\theta = 1$, α must be real. Consequently the input impedance has the form

$$Z = \frac{A + j\sin\theta B}{D + j\sin\theta C}$$

with $C = B$. Thus the impedance matrix has the form

$$\begin{pmatrix} A/R & j\sin\theta B \\ j\sin\theta B/R & D \end{pmatrix}. \quad (15)$$

The important point is that the matrix is symmetric, except for the factor R, and this is the necessary and sufficient condition for the result. Consider

$$\begin{pmatrix} \cos\theta & -j\sin\theta Z_1 \\ -j\dfrac{\sin\theta}{Z_1} & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} A/R & j\sin\theta B \\ j\sin\theta B/R & D \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -j\sin\theta Z_n \\ -j\dfrac{\sin\theta}{Z_n} & \cos\theta \end{pmatrix}. \quad (16)$$

If one considers the term in the first row and column of the left hand product, Z_1 must be chosen so that the coefficient of the highest power of $\cos\theta$ vanishes. Thus Z_1 is equal to the ratio of the highest coefficient of A divided by the leading coefficient in B. Thus $Z_1 = R/Z_n$, if one applies the same argument to the right hand product. If one evaluates the diagonal elements in the triple product, it is found that they have the same type of symmetry as is shown in (15). Thus the same arguments may be repeated with the general consequence that $Z_i Z_{n+1-i} = R$.

It should be observed that, if one assumes symmetry about the other diagonal so that $A = D$, one is led to insertion loss functions of the form

$$P_L = 1 + \sin^2 \theta Q_{n-1}^2(\cos \theta)$$

and one has end-for-end symmetry of the characteristic impedances, and the other properties of a band-pass filter.

We conclude the general discussion of the problem by exhibiting the bi-rational relationships between the coefficients in the impedance function, and the characteristic impedances. Consider the elementary symmetric functions σ_i of the n characteristic impedances $Z_1, \cdots, Z_n$, defined by the sums

$$\sigma_i = \sum Z_j \cdots Z_k$$

where summation is taken over all combinations of the n characteristic impedances taken i at a time. It is further assumed that the Z_i's in each term are ordered so that the index of any Z_i in a term is less than the indices of all the Z_i's which follow it, in same term. This is so-called lexicographical ordering. Then it may be shown by induction that the σ's of (5) may be constructed from the σ_i's as follows:

σ_i^e is constructed from σ_i by replacing alternate Z_i's, starting with the second, in each term by the reciprocal $1/Z_i$.

σ_i^0 is constructed as above except one starts with the first Z_i in each term.

For example for $n = 3$, $\sigma_2^0 = Z_3/Z_1 + Z_3/Z_2 + Z_2/Z_1$ and $\sigma_3^e = Z_1 Z_3 / Z_2$.

For the reverse of this procedure, it is convenient to assume that the input impedance is known in the form of the impedance matrix (2). One will find, in general, that

$$Z_{k+1} = \alpha_{k+1}/\beta_{k+1} \qquad (17)$$

where

$$\alpha_{k+1} = \frac{\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 \\ a_0 & -c_1 & 0 & 0 & \cdots & 0 \\ a_2 & c_1 - c_3 & a_0 & -c_1 & \cdots & 0 \\ \cdot & \cdot & a_2 & c_1 - c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{2k} & c_{2k+1} - c_{2k-1} & \cdot & \cdot & \cdots & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ a_0 & -c_1 & 0 & 0 & \cdots & 0 \\ a_2 & c_1 - c_3 & a_0 & -c_1 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{2k} & c_{2k+1} - c_{2k-1} & \cdot & \cdot & \cdots & -1 \end{vmatrix}}$$

and

$$\beta_{k-1} = \frac{\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 \\ c_1 & -a_0 & 0 & 0 & \cdots & 0 \\ c_3 & -a_2 & c_1 & -a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ c_{2k+1} & -a_{2k} & \cdot & \cdot & \cdots & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ c_1 & -a_0 & 0 & 0 & \cdots & \cdot \\ c_3 & -a_2 & c_1 & -a_0 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{2k+1} & -a_{2k} & \cdot & \cdot & \cdots & -1 \end{vmatrix}}.$$

These determinations are obtained by extending the idea shown in the left hand product of (16). The first impedances $Z_1, \cdots, Z_k$ are assumed known, and the product of the corresponding inverse impedance transformations is written in general form with the coefficients of the elements in the first row given the general designations, $X_1, \cdots, X_k$. When this inverse matrix of degree k in $\cos \theta$ is multiplied by the given impedance matrix, a matrix of degree $k+n$ in $\cos \theta$ results. All the coefficients of $\cos \theta$ in the first row and column of this matrix must be zero, except the coefficient, α_{k+1}, of $\cos^{n-k} \theta$, and the coefficients of lower degree terms. If we treat α_{k+1} as an unknown, we find that we have exactly $k+1$ linear equations for the determination of $k+1$ unknowns, $X_1, \cdots, X_k$ and α_{k+1}. The use of Cramer's rule yields α_{k+1} in determinant form. If the same procedure is applied to $Y_1, \cdots, Y_k$, the coefficients in the 2nd row of the inverse determinant, then β_{k+1}, the highest nonzero coefficient of the lower left hand corner element of the product matrix, can also be determined. The value of Z_{k+1} follows immediately.

Numerical Example

As an illustrative example, consider the problem of designing a transformer from characteristic impedance unity to characteristic impedance 0.440 in waveguide having a cutoff wavelength of 5.680 inches, subject to the requirement that vswr $\leqq 1.05$ from 2600–3600 mc.

It is readily determined that this requires that P_L shall have 3 roots in the pass band, and we select an insertion loss function of the form

$$P_L = 1 + k^2 T_3^2 \left[\frac{\cos \theta}{s} \right]$$

with $\theta = 2\pi l / \lambda_g$. If l is to be chosen so that $\cos \theta = 0$ at the center of the band and of equal magnitude at the ends of the band

$$l = \frac{\lambda_{g1} \lambda_{g2}}{2(\lambda_{g1} + \lambda_{g2})}$$

where λ_{g1} and λ_{g2} are the extreme guide wavelengths. s is chosen by the requirement that the argument of $T_3(\cos\theta/s)$ shall be unity at the ends of the band, and is found to be numerically equal to 0.464. k is determined by the requirement that, at infinite frequency, the transformer shrinks to zero length and P_L is that resulting from the terminating impedance R. In general

$$\frac{(R+1)^2}{4R} = 1 + k^2 T_3^2(1/s)$$

and in our example $k^2 = 1.5777 \times 10^{-4}$. Now

$$|\Gamma|^2 = \frac{P_L - 1}{P_L} = \frac{k^2 T_3^2\left(\frac{\cos\theta}{s}\right)}{0.253\cos^6\theta - 0.0817\cos^4\theta + 0.00659\cos^2\theta + 1}.$$

If, in the denominator, we make the substitution $p = -j\cos\theta/\sin\theta$, its p roots are found to be

$$p_1 = \pm(1.034 + j0.351)$$
$$p_2 = \pm 0.772$$
$$p_3 = \pm(1.034 - j0.351).$$

If we select those roots lying in the left half plane, the denominator of Γ, except for a multiplicative constant, is $p^3 + 2.841p^2 + 2.790p + 0.921$. We may then write

$$\Gamma = \frac{\alpha k T_3(\cos\theta/s)(p^2 - 1)^{3/2}}{p^3 + 2.841p^2 + 2.790p + 0.921}.$$

The multiplicative constant α is chosen by the requirement that $\Gamma = R - 1/R - 1$ when $\cos\theta = 1$ and $p = \infty$. Then Γ may be written

$$\Gamma = \frac{0.921\cos\theta(0.503\cos^2\theta - 0.0812)}{\cos\theta(-3.790\cos^2\theta + 2.790) - j\sin\theta(3.762\cos^2\theta + 0.921)}$$

and the input impedance is

$$Z = \frac{\cos\theta(-3.327\cos^2\theta + 2.716) + j\sin\theta(-3.762\cos^2\theta + 0.921)}{\cos\theta(-4.254\cos^2\theta + 2.865) + j\sin\theta(-3.762\cos^2\theta + 0.921)}.$$

The fact that the imaginary parts of the numerator and denominator of Z are equal assures us of a transformer having symmetrical reflection coefficients as previously proven. The impedance matrix giving rise to this input impedance can be written

$$\begin{pmatrix} \frac{\cos\theta}{R_L}(-3.327\cos^2\theta + 2.716) & j\sin\theta(-3.762\cos^2\theta + 0.921) \\ \frac{j\sin\theta}{R_L}(-3.762\cos^2\theta + 0.921) & \cos\theta(-4.254\cos^2\theta + 2.865) \end{pmatrix}$$

except for a multiplicative factor in each term. Now R_L and the multiplicative factor are uniquely determined by the condition that the impedance matrix must reduce to the unit matrix for $\cos\theta = 1$. Thus the impedance matrix becomes

$$\begin{bmatrix} \cos\theta(5.442\cos^2\theta - 4.442) & j\sin\theta(2.710\cos^2\theta - 0.664) \\ j\sin\theta(6.154\cos^2\theta - 1.507) & \cos\theta(3.064\cos^2\theta - 2.064) \end{bmatrix}.$$

If this is multiplied on the left by the inverse of (1)

$$\begin{pmatrix} \cos\theta & -j\sin\theta Z_1 \\ \frac{-j\sin\theta}{Z_1} & \cos\theta \end{pmatrix}$$

4 equivalent conditions result that the powers of $\cos\theta$ occurring in the result will be at most of second degree. For the coefficient of $\cos^4\theta$ appearing in the upper left hand element, one obtains the coefficient $5.442 - 6.154Z_1$, so that $Z_1 = 0.884$. When this transformer element is removed, the impedance matrix for the balance of the transformer is

$$\begin{pmatrix} 2.34\cos^2\theta - 1.335 & j\sin\theta(1.17\cos\theta) \\ j\sin\theta(3.52\cos\theta) & 1.74\cos^2\theta - 0.749 \end{pmatrix}.$$

The next step yields $Z_2 = 0.668$ and a remaining transformer having an impedance matrix

$$\begin{pmatrix} 0.998\cos\theta & -0.498j\sin\theta \\ 2.01j\sin\theta & 1.02\cos\theta \end{pmatrix}$$

corresponding to an impedance of 0.495. The last step will result in the unit matrix and provides a check on the accuracy of the numerical calculations.

Conclusion

A general solution of the synthesis of equal length impedance transformers with given insertion loss function has been obtained. It shows that optimum Tchebycheff characteristics can be physically realized and are true optimums for quarter wavelength transformers. Collin's conjecture regarding the symmetry of the optimum transformer is proved and closed expressions are given for the bi-rational transformations relating the characteristic impedances to the coefficients occurring in the impedance functions. Finally it is shown that the determination of the characteristic impedances for Tchebycheff or maximally flat performance involves at most the solution of quadratic equations.

Appendix

We wish to prove that a rational function of p with real coefficients, of degree n, in numerator and denominator written

$$Z = \frac{m_1(p) + n_1(p)}{m_2(p) + n_2(p)} \tag{20}$$

with m_1 and m_2 either even or odd and then n_1 and n_2 odd or even is the impedance function of a cascade of equal-length impedance transformers terminated in a resistance if 1) Z is positive real; and 2)

$$m_1(p)m_2(p) - n_1(p)n_2(p) = C(p^2 - 1)^n.$$

This will have been shown, if we can demonstrate the existence of a positive real number Z_c such that the transformation, inverse to (4),

$$Z = Z_c \frac{pZ - Z_c}{-Z + pZ_c} \tag{21}$$

when applied to (20), results in a rational function of one lower degree, with real coefficients for which conditions (1) and (2) are again satisfied. Repetition of this procedure must result in an impedance function which is a positive constant; and, at this point, the synthesis is complete.

Now the general possibility of this reduction is the result of the theorem of Richards previously referred to, with the addition of condition (2). What is required by Richards is that $Z(-1) = -Z(1)$. For the reactance functions considered by Richards, this is almost trivially true at each stage of the synthesis, and is readily derived from condition (2) for the case of resistive terminations as follows

$$Z(1) = \frac{m_1(1) + n_1(1)}{m_2(1) + n_2(1)}$$

but from 2, $m_1(1)m_2(1) = n_1(1)n_2(1)$. Therefore

$$-\frac{m_1(-1)}{n_2(-1)} = \frac{m_1(1)}{n_2(1)} = \frac{n_1(1)}{m_2(1)} = -\frac{n_1(-1)}{m_2(-1)} = r$$

since $n_i(p)/m_i(p)$ is certainly odd. Thus

$$Z(1) = r = -Z(-1).$$

We may now prove, after Richards, that, if we select Z_c to have the value of Z at $p=1$, Z' will be of one less degree in p and be positive real. In the first place, the numerator of Z' vanishes for $p=1$ and hence contains $p=1$ as a factor. It also vanishes for $p=-1$, since $Z(1) = -Z(-1)$, and so contains a factor p^2-1. The same statements are true for the denominator of Z', and thus it too contains a factor p^2-1 which may be cancelled out with the same factor in the numerator. Thus Z' is a rational function of p whose numerator and denominator are of exactly one less degree than the numerator and denominator of Z.

It is pr. Consider the function $(Z'-1)/(Z'+1)$. Clearly a necessary and sufficient condition that Z' be pr is that $|(Z-'1)/(Z'+1)| \leqq 1$ for the Re $(p) \geqq 0$. This is apparent on consideration of the magnitude of the vectors $Z'-1$ and $Z'+1$. Now

$$\frac{Z'/Z(1) - 1}{Z'/Z(1) + 1} = \frac{Z - Z(1)}{Z + Z(1)} \frac{p+1}{p-1}. \tag{22}$$

In the first place, the function on the left is analytic in the right half plane, since Z is known to be pr so that the only zero of the denominator of (22) must occur at $p=1$ where it is cancelled out by a corresponding zero in $Z-Z(1)$. Now it is well known (maximum modulus theorem again) that an analytic function assumes the maximum value for its absolute values on the boundary of its region of analyticity. Thus the absolute value of (22) in the right half plane is less than or equal to the minimum absolute value assumed by it on the imaginary axis. Here $|(p+1)/(p-1)| = 1$. Moreover since Z is assumed pr, $|[Z-Z(1)]/[Z+Z(1)]| \leqq 1$. Thus

$$\left|\frac{Z'/Z(1) - 1}{Z'/Z(1) + 1}\right| \leqq \left|\frac{Z - Z(1)}{Z + Z(1)}\right| \cdot \left|\frac{p+1}{p-1}\right| \leqq 1$$

on the imaginary and hence everywhere in the right half plane. Accordingly $Z'/Z(1)$ is pr, and then so is Z' since $Z(1)$ is positive.

The proof of the synthesis theorem will be completed when we argue that for Z' written in the form

$$(m_1'(p) + n_1'(p))/(m_2'(p) + n_2'(p)),$$

$$m_1'(p)m_2'(p) - n_1'(p)n_2'(p) = C(p^2 - 1)^{n-1}.$$

Consideration of the inverse transformation (22) shows that the corresponding impedance matrix has the determinant, p^2-1, just as the transformation (4). Before any cancellation then

$$m_1'(p)m_2'(p) - n_1'(p)n_2'(p) = C(p^2 - 1)^{n+1}.$$

When $Z_c \to Z(1)$, Richards' theorem shows that a term p^2-1 factors from both numerator and denominator. Actually p^2-1 must factor from both even and odd parts of numerator and denominator, and when this is done the result follows.

Discussion

H. Seidel in a doctoral dissertation dated May, 1954 has shown that an exact synthesis can be carried through, starting from an insertion loss function of allowed form. His procedure parallels that used in our numerical example. He does not introduce a complex variable equivalent to p, however, and thus does not have Richards' theorem available for proving physical realizability. In particular, he makes no point of the second condition for the physical realizability of an impedance function. Ozaki and Ishii,[11] clearly state this second condition, but they do not parallel Darlington by starting from a given insertion loss function. E.M.T. Jones in the 1956 IRE CONVENTION RECORD uses a complex variable, but he makes no mention of the second condition for physical realizability, and appears, in his proof of physical realizability, to have appealed to Richards for a theorem which Richards did not prove.

[11] H. Ozaki and J. Ishii, "Synthesis of transmission-line networks and the design of uhf filters," IRE TRANS., vol. CT-2, p. 325-336; December, 1955.

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An Analysis of the Diode Mixer Consisting of Nonlinear Capacitance and Conductance and Ohmic Spreading Resistance*

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Summary—A method is presented for calculating the mixer admittance matrix Y' which results when an ohmic impedance is connected in series with a diode mixer described by an admittance matrix Y. There are no restrictions on the frequency dependence of the ohmic impedance nor on the number of harmonic sidebands considered. The equations are worked out in detail for the "low Q" case in which signal, image, and intermediate frequencies are considered, and it is shown that Y' in this case is "nearly low Q." As a result of this analysis the usual criterion for good high-frequency mixing, *i.e.*, that the product of the spreading resistance and the barrier capacitance be small compared with unity, is criticized and a new figure of merit is proposed.

Explicit formulas have been derived for calculating the elements of Y' when Y represents the parallel combination of a nonlinear conductance and capacitance. In general, these formulas are cumbersome, but three special cases have been considered in detail.

Case 1: Zero spreading resistance and equal admittances connected to image and signal terminals. Results: a) The conversion gain is independent of the contact area. b) Regions of negative IF conductance are always associated with arbitrarily high gain.

Case 2: High-frequency, small spreading resistance, image shorted across nonlinear conductance and capacitance. Results: a) The conversion loss and the IF admittance can be given by closed equations. b) The IF conductance can be negative. c) Regions of negative IF conductance are bounded by regions of arbitrarily small IF conductance. d) The conversion loss can *decrease* with increasing frequency. e) Low conversion loss is accompanied by narrow bandwidth.

Case 3: The spreading resistance is zero and the image is shorted. Results: a) Above a certain frequency negative IF conductance is obtained and arbitrarily low conversion loss is possible. b) The situation is quite similar to that of Case 1.

Measurements of mixer performance at the "available terminals" are discussed and the failure of the "phenomenological theory of mixing" as a basis for making such measurements is emphasized.

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INTRODUCTION

THIS PAPER will be concerned principally with the mixing properties of the circuit of Fig. 1 (next page), where arrows indicate that g and C are functions of the voltage across them. Frequent reference will be made to Torrey and Whitmer[1] and whenever possible the notation used therein will be followed here.

The circuit of Fig. 1 has been widely used, qualitatively at least, as an equivalent circuit for point-contact crystal diodes,[2] particularly for microwave work in which the capacitor is of importance. The part of the crystal diode that Fig. 1 is supposed to represent is shown in Fig. 2. The terminals are at the dotted lines AA' and BB'. The distance from the line AA' to the surface is a small fraction of the shortest wavelength involved, while the line BB' is located so as to include nearly all of the spreading resistance. It can be shown that the latter requirement will be fulfilled if BB' is several times the contact diameter away from the contact region.

The validity of the circuit of Fig. 1 as a representation of Fig. 2 is open to question. It has been verified in the

[1] H. C. Torrey and C. A. Whitmer, "Crystal Rectifiers," McGraw-Hill Book Co., Inc., New York, N. Y.; 1948.
[2] *Ibid.*, p. 24.